viscosity coefficient of the fluid be larger than this eddy viscosity coefficient, then the flow is relatively viscous already, and the problem of smaller scales of motion does not arise.

For equilibrium spectra of turbulence in a range of scales at which negligible molecular viscous dissipation is occurring, ϵ is independent of λ, and the eddy viscosity coefficient is proportional to $\lambda^{4/3}$. This relationship has abundant observational verification in the atmosphere and ocean with values of α of the order of 1.

In a numerical model we may take the scale λ to be of the order of the grid distance Δx. There remains the problem of determining ϵ. A consistent estimate of ϵ can be based on the dissipation of energy from the large scales as calculated, namely, $\epsilon = \nu D^2$. Here D^2 is a finite difference approximation to the positive scalar $2s^{ij}s_{ij}$ where

$$s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

is the rate of strain tensor defined in terms of the derivatives of the velocity $u_{i,j}$. We are assuming here for simplicity, but somewhat arbitrarily, that the bulk coefficient of eddy viscosity is $\frac{2}{3}$ of the shear coefficient, i.e., that the second coefficient is zero. We are also assuming isotropic eddies in defining a single eddy viscosity coefficient. Both assumptions are open to question. Then

$$\nu = \alpha \nu^{1/3} D^{2/3} \lambda^{4/3}$$

or

$$\nu = \alpha^{3/2} D \lambda^2$$

The viscous stress tensor is given by

$$\sigma^{ij} = 2\nu\rho s^{ij} = 2\alpha^{3/2}\lambda^2\rho[2s^{mn}s_{mn}]^{1/2}s^{ij}$$

where, of course, finite difference approximations must be made.

For flow in one space dimension, the arguments about small eddies seem to make no sense; however, the formation of shocks has much the same formal character of a cascading of energy from larger to smaller scales of motion as a consequence of the nonlinear nature of the equations. If we specialize the preceding results to this case, then in finite difference approximation

$$s_{11} = \Delta u/\Delta x \qquad D = 2^{1/2}|\Delta u/\Delta x|$$

$$\nu = \alpha^{3/2}2^{1/2}|\Delta u|\Delta x \qquad \sigma^{11} = (2\alpha)^{3/2}\rho|\Delta u|\Delta u$$

which leads to a viscous pressure $q = -\sigma^{11}$. This is the form of nonlinear artificial viscosity proposed in 1950 by von Neumann and Richtmyer[2] and used since in many shock hydrodynamics calculations. It would not have come from the more general formulation if the bulk coefficient of eddy viscosity had been set to zero. We see, incidentally, that the eddy viscosity coefficient has the form $\nu = l^2|u_{i,j}|$ given by Prandtl's mixing-length theory with the mixing length $l \approx \lambda \approx \Delta x$.

Finally we can recognize limitations in obtaining numerical solutions of the Navier-Stokes equations when molecular viscosity is to be large enough so that we need not introduce an eddy viscosity. If n is the number of grid points in one direction, we can estimate

$$\Delta u \approx U/n \qquad \Delta x \approx L/n$$

where U, L are characteristic velocities and lengths in the large for the flow. Then $R = UL/\nu \approx n^2$ is the largest Reynolds number for which no eddy viscosity need be introduced. Since, for three-dimensional problems, the computing time is proportional to n^4, we see that the maximum Reynolds number of proper three-dimensional Navier-Stokes computations is going to increase only as the square root of available computing speeds.

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Magnetohydrodynamic-Hypersonic Viscous and Inviscid Flow near the Stagnation Point of a Blunt Body

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In studies of magnetohydrodynamic re-entry phenomena, the interaction between the magnetic field carried by the re-entry vehicle and the flow of the partially ionized gas surrounding the vehicle is given by the magnetic interaction parameter. This parameter is the product of the magnetic Reynolds number and the ratio of the magnetic pressures to the dynamic pressures. For a small magnetic interaction parameter the flow is essentially undisturbed by the magnetic field, and the induced magnetic field is negligible in comparison with the primary field. The magnetic field at any point in the flow can be assumed to be that of the primary field. By increasing the magnetic interaction parameter, i.e., increasing the conductivity (magnetic Reynolds number) or the applied magnetic field, the flow is no longer undisturbed. The effect, which is of concern here, is that the induced magnetic field is no longer negligible in comparison with the primary field.

An approach to simplifying the involved partial differential equations describing the hypersonic flow is to restrict the investigation to a local-similarity solution. For further computational simplification, the magnetic-field strength at the shock usually is assumed as a boundary condition. Using this approach, Bush found that numerical integration becomes impossible for the inviscid case when the value of the magnetic interaction parameter exceeds a certain critical value.[1,2] More recently, a similar result was found by Smith and Wu for the viscous case.[3]

In the present work, the magnetic field is defined at the body where it is a natural characteristic of the problem. The solution then consists of solving ordinary differential equations by a quasi-linearization technique, which simultaneously satisfies boundary conditions on the shock and on the body.

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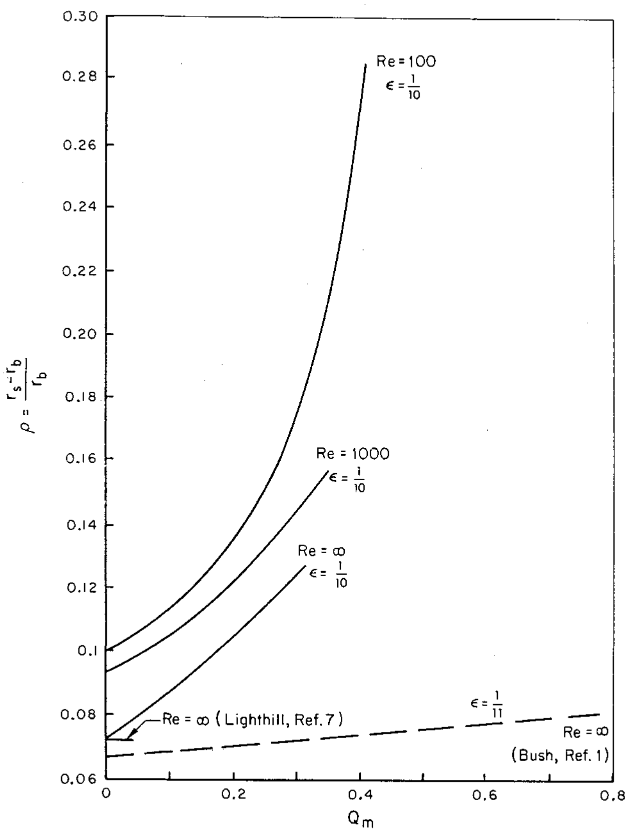

Fig. 1 Standoff distance vs magnetic interaction parameter.

Local-Similarity Solution

The model used here is the same as that used previously in the literature, i.e., a steady-state, constant-density, viscous-layer model. In the stagnation region, the equations of magnetohydrodynamics (MHD), describing the flow, can be written in the form[4]

$$\left.\begin{aligned} \nabla\cdot\mathbf{v} &= 0 \\ \nabla\times[(\nabla\times\mathbf{v})\times\mathbf{v} - (\mu_e/4\pi\rho)(\nabla\times & \\ \mathbf{H})\times\mathbf{H}] &= \nu\nabla^2\nabla\times\mathbf{v} \\ \nabla\cdot\mathbf{H} &= 0 \\ \nabla\times[\nabla\times\mathbf{H} - 4\pi\sigma\mu_e\mathbf{v}\times\mathbf{H}] &= 0 \end{aligned}\right\} \qquad (1)$$

where μ_e = magnetic permeability, σ = electrical conductivity, $\mathbf{H}$ = magnetic-field intensity, ν = kinematic viscosity, ∇^2 = Laplacian operator, $\mathbf{v}$ = velocity vector, and ρ = density.

The spherical body is assumed to contain a dipole magnetic field axisymmetric with the flow field. The postulated similarity solutions associated with the stagnation point flow have the following functional form:

$$\left.\begin{aligned} v_r &= v_\infty[2f(\eta)/\eta^2]\cos\theta \\ v_\theta &= -v_\infty[(1/\eta)(df/d\eta)]\sin\theta \\ H_r &= H_0[2g(\eta)/\eta^2]\cos\theta \\ H_\theta &= -H_0[(1/\eta)(dg/d\eta)]\sin\theta \end{aligned}\right\} \qquad (2)$$

where the angle θ is measured from the axis of symmetry, v_∞ and $H_0 = B/r_b{}^3$ are the freestream velocity and reference dipole magnetic field at the body, respectively, and $\eta = r/r_b$ (r_b = body radius).

From Eqs. (1) and (2), using a right-handed spherical coordinate frame with $2\sin\theta \simeq \sin 2\theta$, a system of ordinary differential equations can be found. The equation of motion reduces to

$$f\frac{d}{d\eta}\left[\frac{1}{\eta^2}\left(\frac{2f}{\eta^2} - \frac{d^2f}{d\eta^2}\right)\right] - M_p g\frac{d}{d\eta}\left[\frac{1}{\eta^2}\left(\frac{2g}{\eta^2} - \frac{d^2g}{d\eta^2}\right)\right] = \frac{1}{2Re}\left[\left(\frac{1}{\eta}\frac{d}{d\eta}\frac{\eta^2 d}{d\eta} - \frac{2}{\eta}\right)\left(\frac{2f}{\eta^3} - \frac{1}{\eta}\frac{d^2f}{d\eta^2}\right)\right] \qquad (3)$$

and the induction equation becomes

$$\frac{2g}{\eta^2} - \frac{d^2g}{d\eta^2} = 2R_m\left[g\frac{df}{d\eta} - f\frac{dg}{d\eta}\right]\frac{1}{\eta^2} \qquad (4)$$

where $M_p = (\mu_e H_0{}^2/4\pi\rho_\infty v_\infty{}^2)$ = magnetic pressure number, $R_m = 4\pi\sigma\ \mu_e\ v_\infty r_b$ = magnetic Reynolds number, $Q_m = M_p\ R_m$ = magnetic parameter defined at the body, and $Re = v_\infty r_b/\nu$ = Reynolds number.

Boundary Conditions at the Shock

If the constant-density, shock-layer model is simplified further by assuming a thin viscous boundary layer, the boundary conditions of $f(\eta)$ at the shock, with $\eta_s = r_s/r_b$ (r_s = shock radius), are

$$f(\eta_s) = -\epsilon\eta_s{}^2/2 \qquad (df/d\eta)_{\eta_s} = -\eta_s \qquad (5)$$

where $\epsilon = \rho_\infty/\rho$, the ratio of density across the shock. The other boundary condition at the shock is based on the vorticity jump[4]

$$\left(\frac{d^2f}{d\eta^2}\right)_{\eta=\eta_s} = 2 - 2_\epsilon - \frac{1}{\epsilon} - 4\frac{Q_m}{\eta_s{}^2\epsilon}g\left(2_\epsilon\frac{dg}{d\eta} - \frac{g}{\eta}\right)_{\eta=\eta_s} \qquad (6)$$

Boundary Conditions at the Body

The flow is governed at the body by the condition of nonslip and zero radial velocity

$$f(1) = 0 \qquad (df/d\eta)_{\eta=1} = 0 \qquad (7)$$

In the inviscid solution, the nonslip condition is removed. Two additional conditions at the body are based on the continuity of the magnetic field at the body. These are

$$g(1) = 1 \qquad (dg/d\eta)_{\eta=1} = -1 \qquad (8)$$

where $\eta = 1$ at the body.

Table 1 Component ratios of magnetic field at the shock to magnetic field at the body

R_m	M_p	$\frac{r_s - r_b}{r_s}$	$\frac{g(\eta_s)}{\eta_s{}^2}$	$-\frac{1}{\eta_s}\left(\frac{dg}{d\eta}\right)_{\eta_s}$	$(r_b/r_s)^3$
		$Re = 100$			
0.0	0.0	0.1008	0.7497	0.7497	0.7497
1.0	0.1	0.1137	0.7272	0.6311	0.7239
1.0	0.2	0.1344	0.6884	0.5945	0.6850
1.0	0.3	0.1752	0.6195	0.5308	0.6161
1.0	0.4	0.2619	0.5002	0.4249	0.4977
0.1	1.0	0.1136	0.7243	0.7147	0.7240
0.2	1.0	0.1342	0.6860	0.6672	0.6853
0.3	1.0	0.1745	0.6181	0.5915	0.6171
		$Re = 1000$			
0.0	0.0	0.0929	0.7660	0.7660	0.7670
1.0	0.1	0.1054	0.7435	0.6463	0.7402
1.0	0.2	0.1226	0.7103	0.6146	0.7068
1.0	0.3	0.1449	0.6704	0.5759	0.6667
		$Re = \infty$			
0.0	0.0	0.0731	0.8093	0.8093	0.8094
1.0	0.1	0.0878	0.7799	0.6800	0.7767
1.0	0.2	0.1066	0.7410	0.6420	0.7378
1.0	0.3	0.1241	0.7074	0.6093	0.7039

Computation Procedure

A quasi-linearization technique was used in the solution of the two-point boundary-value problem. The viscous solution consisted of solving ordinary differential equations [Eqs. (3) and (4)], satisfying simultaneously three conditions on the shock and four conditions on the body. Since the shock standoff distance also is unknown, a transformation to independent variable t

$$\eta = [(r_s/r_b) - 1]\,t + 1 \tag{9}$$

gives the integration range from the border to shock $0 \leq t \leq 1$, where the standoff distance $\zeta[\zeta = (r_s/r_b) - 1]$ is an unknown. The differential equation

$$d\zeta/dt = 0 \tag{10}$$

is added to Eqs. (3) and (4). Equations (3, 4, and 9) reduce to a set of seven first-order ordinary differential equations and seven boundary conditions in the viscous solution. The initial value of ζ is determined by the boundary conditions.

In conjunction with the quasi-linearization scheme, it was found necessary to reorthogonalize at every other integration step. A description of the quasi-linearization scheme is found in Ref. 5. The Gram-Schmidt orthogonalization used is described in Ref. 6.

Discussion of Results

The viscous Reynolds numbers *Re* used were 100, 1000, and ∞. The range of magnetic interaction parameters where solutions were possible was rather limited. At the larger magnetic interaction parameters, the quasi-linearization solution was not a solution of the original nonlinear differential equations and was not included. However, a number of conclusions can be drawn from the solutions that were obtained.

The numerical calculations of standoff distance as a function of interaction parameter are illustrated in Fig. 1. These results are in general agreement with previous works; i.e., the shock-wave standoff distance increases with increasing magnetic interaction parameter for a given viscous Reynolds number, and, for a constant magnetic interaction parameter, the standoff distance decreases with increasing viscous Reynolds number, as expected by physical reasoning.[3,4] There are possibly two features of this plot which are new. In previous works, a critical value of the interaction parameter appears at which the shock standoff distance recedes to infinity asymptotically as the applied magnetic field is increased.[3] The critical interaction parameter in the present plot no longer appears as in Ref. 3. Secondly, for $Q_m < 1$, the standoff distance increases much more rapidly for increasing Q_m in this solution than in the work by Bush.[1] If these results are applicable, it would indicate that the magnetic field, even for $Q_m < 1$, is a more effective method of controlling the hypersonic flow than previously thought.

Table 1 gives the computed values of the ratio of the resultant magnetic-field components at the shock to the dipole field at the body and also the ratio of the undistorted dipole field at the shock to that at the body. It is evident from this table that the standoff distance depends essentially upon the product of R_m and M_p, i.e., Q_m. A similar result has appeared in previous calculations.[1,3,4] One further interesting feature of the tabulated results is that the resultant radial field H_r decreases much as a dipole field, whereas the angular field component H_θ decrease is slightly greater. However, for a fixed Q_m, H_θ depends upon the relative values of R_m and M_p. It appears that the induced magnetic field is influenced mainly by the electrical conductivity. Hence a decrease in the conductivity (the induced field) should result in an H_θ more nearly approximated by a dipole field.

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An MHD Boundary-Layer Compatibility Condition

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Nomenclature

x, y	=	coordinates along and normal to surface
u, v, w	=	velocities within boundary layer
δ	=	boundary-layer thickness
η	=	y/δ, dimensionless distance
f	=	$u/U = a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5 + \ldots$
δ^*	=	$\delta\int_0^1(1 - f)d\eta$, displacement thickness
θ	=	$\delta\int_0^1(1 - f)d\eta$, momentum thickness
U_∞	=	undisturbed freestream velocity
τ_0	=	viscous shear stress
μ	=	fluid viscosity
P	=	pressure
ρ	=	density
σ	=	fluid electrical conductivity, assumed to be small
E	=	electric-field strength in system fixed to the fluid[1]
B_0	=	magnetic-field flux density at surface
Λ	=	$(dU/dx)\delta^2/\nu$, shape factor
Λ_m	=	$U_\infty m\delta^2/\nu$, magnetic-shape factor
mx	=	$\sigma_0 B_0{}^2 x/(\rho u_\infty)$, the magnetic-interaction parameter
R_N	=	$u_\infty x/\nu$, Reynolds number
ν	=	μ/ρ, kinematic viscosity

ORDINARILY, the boundary-layer equations, which represent the second law of motion and conservation of mass, are solved at every point within the boundary layer. In the von Karman-Pohlhausen method, it is assumed that it is sufficiently accurate to satisfy these equations on the average over the boundary-layer thickness. This is done by integrating the equation of motion over the boundary-layer thickness. It is this integral equation, which is satisfied then, rather than the equation of motion itself. The compatibility conditions[2] imposed in such solutions are well known. It will be shown, through the following magnetohydrodynamic (MHD) example, that these conditions are

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